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A linear backward Euler scheme for the saturation equation: Regularity results and consistency☆

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ABSTRACT

We consider a linearization of a numerical scheme for the saturation equation (or porous medium equation) $\frac{\partial S}{\partial t} - \nabla \cdot f(S)\mathbf{u} - \nabla \cdot k(S)\nabla S = 0$, through first order expansions of the fractional function f and the inverse of the function $K(s) = \int_0^s k(\tau)\mathrm{d}\tau$, after a regularization of the porous medium equation. We establish a regularity result for the Continuous Galerkin Method and a regularity result for the linearized scheme analogous to the corresponding nonlinear scheme. We then show that the linearized scheme is consistent with the nonlinear scheme analyzed in a previous work.

1. Introduction

Consider the saturation equation

$$\begin{cases} \dfrac{\partial S}{\partial t} + \nabla \cdot (f(S)\mathbf{u}) - \nabla \cdot (k(S)\nabla S) = Q(S) & \text{on } \Omega \times (0, T_0] \\ (f(S)\mathbf{u} - k(S)\nabla S) \cdot \mathbf{n} = q & \text{on } \partial\Omega \times [0, T_0] \\ S(x, 0) = S^0(x) & \text{on } \Omega, \end{cases} \tag{1.1}$$

obtained from modeling a two-phase immiscible flow through a porous medium [1–4]. The set Ω is a bounded domain of $\mathbb{R}^n$, $n = 2, 3$. In this work, we have in mind $n = 2$ and Ω a polygonal domain.

The unknown S is the saturation of the invading phase. The diffusion coefficient k is the conductivity of the medium and satisfies the following conditions.

$$k(0) = k(1) = 0, \tag{1.2}$$

$$k(\xi) \geq \begin{cases} c_1 \xi^{\mu} & \text{if } 0 \leq \xi \leq \alpha_1 \\ c_2 & \text{if } \alpha_1 < \xi < \alpha_2 \\ c_3(1 - \xi)^{\mu} & \text{if } \alpha_2 \leq \xi \leq 1 \end{cases} \tag{1.3}$$

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where $0 < \alpha_1 < \frac{1}{2} < \alpha_2 < 1$, and $0 < \mu \leq 2$. We define K by

$$K(s) = \int_0^s k(\tau)\mathrm{d}\tau. \tag{1.4}$$

The graph of the fractional flow function f is known to have an S-shaped (as a function of the saturation S). So we impose the following conditions on f.

The function f is twice continuously differentiable in the variable s, and

$$f'(0) = f'(1) = 0. \tag{1.5}$$

We notice, by [5,6], that if (1.3) and (1.5) hold, then

$$|f(s_2) - f(s_1)|^2 \leq C|K(s_2) - K(s_1)||s_2 - s_1|. \tag{1.6}$$

We also notice, through (1.6) and (1.4), that

$$|f'(s)| \leq C\sqrt{k(s)}. \tag{1.7}$$

For the purpose of establishing regularity estimates, we assume that $k \in C^1([1, 0])$ and $f \in C^2([0, 1])$.

As in [5,7,6,8], we assume that the Darcy velocity $\mathbf{u}$ is given and has the necessary regularity we need for this analysis.

The main purpose of this paper is to establish regularity estimates for a linearization of the Backward Euler scheme obtained by fully discretizing a regularization of problem (1.1). These regularity results are then used to show that the proposed linearized scheme is consistent, in some sense, with the nonlinear scheme considered in [8]. Problem (1.1) has been studied by many authors under various conditions (see for instance [9–11,6,5,8,12], among others). In the study of the problem, two difficulties (among others) arise: the nonlinearity and the degeneracy of the problem. Because of the degeneracies ($k(0) = k(1) = 0$), Problem (1.1) has often been regularized into a family of nondegenerate problems whose solutions converge to the solution of (1.1) [6,5,8,10,9,12]. Though some studies (see for instance [13]) bypass the regularization step, usually, the numerical approximation of the solution of (1.1) is done in three steps: regularization, Continuous Galerkin Method, and fully discretized Galerkin method. In the last step, some of the works cited above obtain a nonlinear implicit scheme (backward Euler) which needs to be linearized in some way for a computer implementation. Often, a Picard iteration is used (see for instance [14] and [15]).

This paper is a continuation of [7] where a method was proposed that linearizes the scheme. We wish to replace the nonlinear scheme

$$\left(\frac{H_\beta(U_h^{n+1}) - H_\beta(U_h^n)}{\Delta t}, \chi\right) - \left(f(H_\beta(U_h^{n+1}))\mathbf{u}^{n+1}, \nabla\chi\right) + \left(\nabla U_h^{n+1}, \nabla\chi\right) = 0 \tag{1.8}$$

analyzed in [8], (where U_h^n is a discrete approximation of $K(S)$), by the linear scheme (3.1), proposed in [7], through first order expansions of the functions $f \circ H_\beta$ and H_β, where H_β is defined by (2.15). The Taylor expansions considered are:

$$H_\beta(v_2) - H_\beta(v_1) = (v_2 - v_1)H'_\beta(v_1) + O((v_2 - v_1)^2),$$

and

$$(f \circ H_\beta)(v_2) - (f \circ H_\beta)(v_1) = (v_2 - v_1)(f \circ H_\beta)'(v_1) + O((v_2 - v_1)^2).$$

The linearized scheme (3.1) and (3.2), below, is obtained from (1.8), by simply discarding the second and higher order terms in the expansions of H_β and $f \circ H_\beta$.

Using one of the regularity results established here, we show that, in fact, if $v_2 = U_h^{n+1}$ and $v_1 = U_h^n$, then

$$O((v_2 - v_1)^2) = O((\Delta t)^\alpha), \tag{1.9}$$

for some $\alpha > 0$, where the constants intervening in the above estimate are independent of β, the regularization parameter, h, the space discretization parameter, and Δt, the time stepping parameter, though this is true for a balanced choice of β, h, and Δt.

The remainder of the paper is structured as follow.

In Section 2, we state some preliminary results established in previous works.

In Section 3, we state and prove our main results: We prove a new regularity result for the Continuous Galerkin Method and use it to establish the consistency of the linearized scheme. We also prove, for the linearized scheme, a regularity result analogue to some regularity results for the Continuous Galerkin Method and the fully discretized nonlinear scheme established in [8].

Notations. Finally, we set additional notation which will be used throughout the remainder of this paper. We define $(f, g) := (f, g)_\Omega := \int_\Omega fg\mathrm{d}x$ when this has a meaning. The notation $\|f\|_{L^p} := \|f\|_{L^p(\Omega)}$ is used for the standard Lebesgue norm of a measurable function, when this quantity is finite. Similarly, we denote by $\|f\|_{L^p(L^q)} := \|f\|_{L^p(0,T,L^q(\Omega))}$ the mixed Lebesgue norm for f. For a vector $\mathbf{v} = (v_1, v_2, \ldots, v_k)$, by $\|\mathbf{v}\|_{L^2}$, we mean $\|\mathbf{v}\|_{(L^2(\Omega))^k}$. We write $V_{ht} := \frac{\partial V_h}{\partial t}$, the partial derivative of V_h with respect to t. Similarly, V_{htt} designates the second order partial derivative of V_h with respect to t. We use C, c, to denote constants which may change from line, but which are independent of the parameters β, h and Δt, unless otherwise explicitly specified. We also denote by $\sigma,\ \sigma_1,\ \sigma_2, \ldots$ etc, constants we can control thanks to the Arithmetic-Geometric Inequality.

2. Preliminary results

In this section, we summarize previous results that are useful for the present analysis.

2.1. The regularized problem

As stated above, because of the degeneracy, Problem (1.1) is often regularized to obtain a nondegenerate problem to which one applies a numerical approximation method. As in [6,5,8,9], we simply perturb the diffusion coefficient k to k_β in such a way that k_β converges strongly to k as the perturbation parameter β tends to 0.

For example, let $0 < \beta \le \frac{1}{2}$ and define k_β by

$$k_\beta(s) = \max(k(s), c_0\beta^\mu) \tag{2.1}$$

where μ is as in (1.3). Round the corners as needed for regularity. Then, obviously,

$$k_\beta(s) \ge c_0\beta^\mu > 0. \tag{2.2}$$

Another possible perturbation is as follows: Let

$$\delta = \min(k(\beta), k(1-\beta)). \tag{2.3}$$

Define k_β by

$$\begin{cases} k_\beta(s) = k(s) & \text{if } k(s) \ge \delta \\ \dfrac{1}{2}\delta \le k_\beta(s) \le \delta & \text{otherwise.} \end{cases} \tag{2.4}$$

Round the corners as needed for regularity. Then $k_\beta(t) \ge k(t)$, for all $t \in [0, 1]$. Also k_β satisfies

$$k_\beta(s) \ge \frac{1}{2}\delta, \quad \forall s \in [0, 1]. \tag{2.5}$$

Thus k_β is bounded away from 0, for fixed β.

Define

$$K_\beta(s) = \int_0^s k_\beta(\tau)\mathrm{d}\tau. \tag{2.6}$$

Replace k by k_β in the original problem to get the new, now nondegenerate, problem:

$$\begin{cases} \dfrac{\partial S_\beta}{\partial t} + \nabla\cdot(f(S_\beta)\mathbf{u}) - \nabla\cdot(k_\beta(S_\beta)\nabla S_\beta) = Q(S_\beta) & \text{on } \Omega\times(0, T_0] \\ (f(S_\beta)\mathbf{u} - k(S_\beta)\nabla S_\beta)\cdot\mathbf{n} = q & \text{on } \partial\Omega\times[0, T_0] \\ S_\beta(x, 0) = S^0(x) & \text{on } \Omega. \end{cases} \tag{2.7}$$

For the remaining of this paper, we assume, to simplify, that $Q \equiv 0$ and $q \equiv 0$.

Set

$$C_0(\beta) = \|K_\beta(\cdot) - K(\cdot)\|_{L^\infty}^\gamma, \tag{2.8}$$

where

$$\gamma = \frac{2+\mu}{1+\mu} \tag{2.9}$$

is the conjugate of $2 + \mu$.

Then, by [6], we have

$$\|K_\beta(S_\beta) - K(S)\|_{L^2(L^2)}^2 = O(C_0(\beta)). \tag{2.10}$$

2.2. Continuous Galerkin method

Let $\{M_h\}_{0<h<1}$ be a family of finite dimensional spaces, with $M_h \subset H^1(\Omega)$, and assume that M_h has the approximation property:

$$\inf_{\chi\in M_h}\|f-\chi\|_{L^p(\Omega)} \le Ch^2|f|_{W^{2,p}} \quad \text{for all } f \in W^{2,p}(\Omega). \tag{2.11}$$

We will also need the *inverse estimate* assumption on M_h (see, for example, Section 4.5 of [16]):

$$\|\chi\|_{H^1} \leq Ch^{-1}\|\chi\|_{L^2} \quad \text{for all } \chi \in M_h. \tag{2.12}$$

To account for possible numerical oscillations, extend k_β as follows (and call it again k_β):

$$k_\beta(\xi) = \begin{cases} k_\beta(1) & \text{if } \xi \geq 1 \\ k_\beta(-\xi) & \text{if } \xi \leq 0. \end{cases} \tag{2.13}$$

For the same reason, extend the fractional function f as follows.

$$f(\xi) = \begin{cases} f(1) & \text{if } \xi \geq 1 \\ f(-\xi) & \text{if } \xi \leq 0. \end{cases} \tag{2.14}$$

Then K_β is bijective from $\mathbb{R}$ to $\mathbb{R}$. So set

$$H_\beta = K_\beta^{-1}. \tag{2.15}$$

Consider the discretized problem: Find $V_h \in M_h$ such that

$$\left(\frac{\partial}{\partial t}H_\beta(V_h), \chi\right) - (f(H_\beta(V_h))\mathbf{u}, \nabla\chi) + (\nabla V_h, \nabla\chi) = 0 \tag{2.16}$$

for all $\chi \in M_h$, and $t \in (0, T_0]$ with the initial condition:

$$P_h H_\beta(V_h(0)) = P_h S^0 \tag{2.17}$$

where S^0 is as in (1.1), and P_h the L^2 projection onto M_h. In fact, $V_h = V_{h,\beta}$, but to simplify the notations, we drop the subscript β. V_h is hopefully the Galerkin approximation to $K_\beta(S_\beta)$ with S_β the solution to Problem (1.1). Indeed, by [8], we have

$$\|V_{ht}\|_{L^2(L^2)} \leq C(\mathbf{u}) \tag{2.18}$$

and

$$\|V_h - K_\beta(S_\beta)\|_{L^2(L^2)} \leq Ch^{\frac{2+\mu}{2}\lambda} \tag{2.19}$$

with

$$\lambda = \frac{4+\mu}{2+4\mu+\mu^2}. \tag{2.20}$$

3. Linearization, regularity, and consistency

In this section, we consider the perturbation given by (2.3)–(2.6) and the linearized scheme below, proposed in [7]. The proposed scheme is: Find a sequence of functions $\{U_h^n\}_{n=0}^N$ of M_h verifying

$$\left(\frac{U_h^{n+1} - U_h^n}{\Delta t}H'_\beta(U_h^n), \chi\right) - \left(\left\{f \circ H_\beta(U_h^n) + (U_h^{n+1} - U_h^n) \times (f \circ H_\beta)'(U_h^n)\right\}\mathbf{u}^{n+1}, \nabla\chi\right) + \left(\nabla U_h^{n+1}, \nabla\chi\right) = 0,$$
$$\forall \chi \in M_h \; 0 \leq n \leq N-1 \tag{3.1}$$

$$P_h H_\beta U_h^0 = P_h S^0 \tag{3.2}$$

where H_β is defined by (2.15). Here $\mathbf{u}^n := \mathbf{u}(\cdot, t^n) = \mathbf{u}(\cdot, n\Delta t)$.

We want to show that this scheme is consistent with the nonlinear scheme (1.8) analyzed in [8]. The analysis there showed that the solution yielded by the scheme converged to the solution to the initial problem (1.1). Thus our result will show that the linearized system (3.1) and (3.2) is consistent with the initial problem (1.1).

Let $\mathbf{A}$ be the matrix of the system of linear algebraic equations given by (3.1) and (3.2) (see [7]). The following theorem, which shows the existence and uniqueness for the system above, was proved in [7].

Theorem 3.1. *Let $v \in M_h$. Then under conditions (1.2)–(1.6), we have*

$$({}^t v, \mathbf{A}v) \geq c_2\left(1 - \frac{\sqrt{\Delta t}\,c_3(\mathbf{u})}{2}\right)\|v\|^2_{L^2(\Omega)} + \Delta t\|\nabla v\|^2_{L^2(\Omega)} \tag{3.3}$$

where c_2 and c_3 are independent of β, h, and Δt.

Remark 3.2. The scheme defined by (3.1) and (3.2) will approximate $K_\beta(S_\beta(\cdot,\cdot))$. Since we might not have a close form of H_β, the inverse function of K_β, in order to recover an approximation of $S_\beta(\cdot,\cdot)$, we can use a numerical procedure for solving nonlinear equations. In order to solve the nonlinear equation

$$K_\beta(S_h((x,t^n))) = U_h^n(x)$$

at each grid point (x,t^n), for $S_h(x,t^n)$, the Newton Method would be a good choice, since $K_\beta{}' = k_\beta > 0$. A linear interpolation, for example, of $S_h(x,t^n)$ would then be a an approximation of $S_\beta(x,t)$. We intend to come back to this aspect in the sequel of this work that will deal with the effective numerical computation of the solution of (3.1) and (3.2).

3.1. A regularity result for the continuous Galerkin method

We prove a new regularity result for the Continuous Galerkin Method applied to the saturation Eq. (1.1). This result helps us establish the consistency of the linear scheme (3.1) and (3.2). The result obtained in the next subsection and the present result, by themselves, could motivate this work, as they are extensions of previous estimates for the problem (see [8]).

Lemma 3.3. *Let V_h be the solution to the problem (2.16) and (2.17). Suppose that conditions (1.2)–(1.7) and (2.12) hold. Then*

$$\|V_{ht}\|^2_{L^\infty(L^2)} \le C(\mathbf{u})\delta^{-6}h^{-4}, \tag{3.4}$$

where $C(\mathbf{u}) = C\left(\mathbf{u}, \left\|\sqrt{H'_\beta(V_h^0)}\,\lim_{t\to 0}(V_{ht})\right\|^2_{L^2}\right)$.

Proof. Let $\chi = V_{htt}$ in (2.16). Then we have:

$$\left((H_\beta(V_h))_t, V_{htt}\right) - \left(f(H_\beta(V_h))\mathbf{u}, \nabla V_{htt}\right) + (\nabla V_h, \nabla V_{htt}) = 0. \tag{3.5}$$

We can rewrite the first term on the left of (3.5) as follows:

$$\begin{aligned}\left((H_\beta(V_h))_t, V_{htt}\right) &= \left(\sqrt{H'{}_\beta(V_h)}V_{ht}, \sqrt{H'{}_\beta(V_h)}V_{htt}\right)\\ &= \left(\sqrt{H'{}_\beta(V_h)}V_{ht}, \left(\sqrt{H'{}_\beta(V_h)}V_{ht}\right)_t - \left(\sqrt{H'{}_\beta(V_h)}\right)_t V_{ht}\right)\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{H'{}_\beta(V_h)}V_{ht}\right\|^2_{L^2} - \left(\sqrt{H'{}_\beta(V_h)}V_{ht}, \left(\sqrt{H'{}_\beta(V_h)}\right)_t V_{ht}\right),\end{aligned} \tag{3.6}$$

where we have used the Product Rule. We have also used the Chain Rule: $(H_\beta(V_h))_t = H'_\beta(V_h)V_{ht}$.

The second term of (3.5) is treated as follows:

$$\left(f(H_\beta(V_h))\mathbf{u}, \nabla V_{htt}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(f(H_\beta(V_h))\mathbf{u}, \nabla V_{ht}\right) - \left((f(H_\beta(V_h))\mathbf{u})_t, \nabla V_{ht}\right). \tag{3.7}$$

The third term can be rewritten as follows:

$$(\nabla V_h, \nabla V_{htt}) = \frac{\mathrm{d}}{\mathrm{d}t}(\nabla V_h, \nabla V_{ht}) - \|\nabla V_{ht}\|^2_{L^2}. \tag{3.8}$$

In view of (3.6) through (3.8), (3.5) becomes:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{H'{}_\beta(V_h)}V_{ht}\right\|^2_{L^2} \le{}& \frac{1}{2}\left\|\sqrt{H'{}_\beta(V_h)}V_{ht}\right\|^2_{L^2} + c\|\nabla V_{ht}\|^2_{L^2} + \frac{1}{2}\left\|\left(\sqrt{H'{}_\beta(V_h)}\right)_t V_{ht}\right\|^2_{L^2} - \frac{\mathrm{d}}{\mathrm{d}t}(\nabla V_h, \nabla V_{ht})\\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\left(f(H_\beta(V_h))\mathbf{u}, \nabla V_{ht}\right) + \|(f(H_\beta(V_h))\mathbf{u})_t\|^2_{L^2}.\end{aligned} \tag{3.9}$$

We treat each of the last three terms on the right hand side of (3.9) individually. The third term is treated as follows:

$$\left(\sqrt{H'{}_\beta(V_h)}\right)_t = \frac{1}{2}\frac{H''{}_\beta(V_h)}{\sqrt{H'{}_\beta(V_h)}}V_{ht},$$

by the Chain Rule, so that

$$\left\|\left(\sqrt{H'{}_\beta(V_h)}\right)_t V_{ht}\right\|^2_{L^2} = \frac{1}{4}\left\|\frac{H''{}_\beta(V_h)}{\sqrt{H'{}_\beta(V_h)}}V_{ht}^2\right\|^2_{L^2}. \tag{3.10}$$

Now, by another use of the Chain Rule and the use of (2.6) and (2.15), one sees that

$$H''{}_\beta(V_h) = -\frac{k'{}_\beta(H_\beta(V_h))}{\left(k_\beta(H_\beta(V_h))\right)^3}.$$

Next use the fact that

$$\|v^2\|_{L^2} = \|v\|_{L^4}^2, \tag{3.11}$$

for all $v \in L^4(\Omega)$, and (2.5) to see, from (3.10), that

$$\left\|\left(\sqrt{H'_\beta(V_h)}\right)_t V_{ht}\right\|_{L^2}^2 \le \frac{C}{\delta^6}\|V_{ht}\|_{L^4}^4. \tag{3.12}$$

The last term on the right side of (3.9) can be treated as follows.

$$\begin{aligned}
\|(f(H_\beta(V_h))\mathbf{u})_t\|_{L^2} &= \|(f\circ H_\beta)'(V_h)V_{ht}\mathbf{u} + f(H_\beta(V_h))\mathbf{u}_t\|_{L^2} \\
&\le \|(f\circ H_\beta)'(V_h)V_{ht}\mathbf{u}\|_{L^2} + \|f(H_\beta(V_h))\mathbf{u}_t\|_{L^2} \\
&\le \left\|\frac{f'(H_\beta(V_h))}{k_\beta(H_\beta(V_h))}V_{ht}\mathbf{u}\right\|_{L^2} + C(\|\mathbf{u}_t\|_{L^2}) \\
&\le \frac{C(\mathbf{u})}{\sqrt{\delta}}\|V_{ht}\|_{L^2} + C(\|\mathbf{u}_t\|_{L^2}),
\end{aligned} \tag{3.13}$$

where we have used (1.7). In view of (3.10)–(3.13), estimate (3.9) becomes

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{H'_\beta(V_h)}V_{ht}\right\|_{L^2}^2 \le{}& \frac{1}{2}\left\|\sqrt{H'_\beta(V_h)}V_{ht}\right\|_{L^2}^2 + c\|\nabla V_{ht}\|_{L^2}^2 + C(\mathbf{u})\left(\frac{1}{\delta^6}\|V_{ht}\|_{L^4}^4 + \frac{1}{\delta}\|V_{ht}\|_{L^2}^2 + C(\|\mathbf{u}_t\|_{L^2})\right) \\
&+ \frac{\mathrm{d}}{\mathrm{d}t}\left(f(H_\beta(V_h))\mathbf{u}, \nabla V_{ht}\right) - \frac{\mathrm{d}}{\mathrm{d}t}(\nabla V_h, \nabla V_{ht}).
\end{aligned} \tag{3.14}$$

Apply the Gronwall Lemma to (3.14) to obtain:

$$\begin{aligned}
\left\|\sqrt{H'_\beta(V_h)}V_{ht}\right\|_{L^\infty(L^2)}^2 \le{}& C(\mathbf{u})\left\{\|\nabla V_{ht}\|_{L^2(L^2)}^2 + \frac{1}{\delta^6}\|V_{ht}\|_{L^4(L^4)}^4 + \frac{1}{\delta}\|V_{ht}\|_{L^2(L^2)}^2 + |(f(H_\beta(V_h))\mathbf{u}, \nabla V_{ht})|_{t=T}\right\} \\
&+ C(\|\mathbf{u}_t\|_{L^2} + |(f(H_\beta(V_h))\mathbf{u}, \nabla V_{ht})|_{t=0}) + c\left(\left\|\sqrt{H'_\beta(V_h)}V_{ht}|_{t=0}\right\|_{L^2}^2 + |(\nabla V_h, \nabla V_{ht})|_{t=T}\right. \\
&\left. + \ |(\nabla V_h, \nabla V_{ht})|_{t=0}\right).
\end{aligned} \tag{3.15}$$

We note that, by the Arithmetic-Geometric Inequality, we have

$$|(f(H_\beta(V_h))\mathbf{u}, \nabla V_{ht})|_{t=T} \le C\frac{h^{-2}}{\sigma_1}\|f(H_\beta(V_h))\mathbf{u}\|_{L^\infty(L^2)}^2 + \frac{\sigma_1}{4}h^2\|\nabla V_{ht}\|_{L^\infty(L^2)}^2, \tag{3.16}$$

and

$$|(f(H_\beta(V_h))\mathbf{u}, \nabla V_{ht})|_{t=0} \le C\frac{h^{-2}}{\sigma_1}\|f(H_\beta(V_h))\mathbf{u}\|_{L^\infty(L^2)}^2 + \frac{\sigma_1}{4}h^2\|\nabla V_{ht}\|_{L^\infty(L^2)}^2, \tag{3.17}$$

where h is the space discretization parameter. In the same manner, we have

$$|(\nabla V_h, \nabla V_{ht})|_{t=T} \le C\frac{h^{-2}}{\sigma_1}\|\nabla V_h\|_{L^\infty(L^2)}^2 + \frac{\sigma_1}{4}h^2\|\nabla V_{ht}\|_{L^\infty(L^2)}^2, \tag{3.18}$$

and

$$|(\nabla V_h, \nabla V_{ht})|_{t=0} \le C\frac{h^{-2}}{\sigma_1}\|\nabla V_h\|_{L^\infty(L^2)}^2 + \frac{\sigma_1}{4}h^2\|\nabla V_{ht}\|_{L^\infty(L^2)}^2. \tag{3.19}$$

The inverse estimate assumption (2.12) yields

$$\|\nabla V_{ht}\|_{L^2} \le ch^{-1}\|V_{ht}\|_{L^2}. \tag{3.20}$$

So,

$$\|\nabla V_{ht}\|_{L^2(L^2)} \le \frac{C(\mathbf{u})}{h}, \tag{3.21}$$

by (2.18) and (3.20), and

$$\|\nabla V_{ht}\|_{L^\infty(L^2)} \le ch^{-1}\|V_{ht}\|_{L^\infty(L^2)}. \tag{3.22}$$

The Sobolev Inclusions state that $H^1(\Omega)$ is continuously embedded in $L^4(\Omega)$, for the spatial dimension $n = 2, 3$ (see, for instance, [17,18]). So estimates (2.18) and (3.20) give:

$$\|V_{ht}\|_{L^4} \le c\|V_{ht}\|_{H^1} \le Ch^{-1}\|V_{ht}\|_{L^2}. \tag{3.23}$$

By Remark 3.1 of [8],

$$\|\nabla V_h\|_{L^\infty(L^2)} \le C(\mathbf{u}). \tag{3.24}$$

Putting estimates (3.16) through (3.24) together, estimate (3.15) becomes

$$\left\|\sqrt{H'_\beta(V_h)}V_{ht}\right\|^2_{L^\infty(L^2)} \le C(\mathbf{u})\delta^{-6}h^{-4} + \sigma_1\|V_{ht}\|^2_{L^\infty(L^2)}. \tag{3.25}$$

Since k_β is continuous on $[0, 1]$, it is bounded on that interval, so is it extension (2.13). Thus $\frac{1}{k_\beta(s)}$ is bounded away from 0, so is $\sqrt{H'_\beta(V_h)}$. Hence, if we choose σ_1 sufficiently small (and we can do so independently of h and β, thanks to the Arithmetic-Geometric Inequality), we can hide the last term on the right side of (3.25) in the left side of (3.25). This ends the proof of the Lemma. $\square$

We show, next, that the linearized scheme (3.1) and (3.2) is consistent with the Continuous Galerkin Method (2.16) and (2.17), not only for fixed regularization parameter β and spatial discretization parameter h, but also for appropriate choices of β and h in terms of the time stepping parameter Δt.

Using first order Taylor expansions of H_β and $f \circ H_\beta$, we get

$$\frac{H_\beta(V_h^{n+1}) - H_\beta(V_h^n)}{\Delta t} = \frac{V_h^{n+1} - V_h^n}{\Delta t}H'_\beta(V_h^n) + \frac{\left(V_h^{n+1} - V_h^n\right)^2}{2\Delta t}H''_\beta(\phi_h^n), \tag{3.26}$$

where $V_h^n := V_h(\cdot, t^n) = V_h(\cdot, n\Delta t)$, and ϕ_h^n between V_h^n and V_h^{n+1}. Similarly, we get

$$(f \circ H_\beta)(V_h^{n+1}) = (f \circ H_\beta)(V_h^n) + (V_h^{n+1} - V_h^n)(f \circ H_\beta)'(V_h^n) + \frac{\left(V_h^{n+1} - V_h^n\right)^2}{2}(f \circ H_\beta)''(\psi_h^n), \tag{3.27}$$

where ψ_h^n is between V_h^n and V_h^{n+1}.

When we discard the respective last terms on the right sides of (3.26) and (3.27) and replace in the nonlinear scheme (1.8), we get the linear scheme (3.1). The following theorem shows that we can do so.

Theorem 3.4. *The linear scheme (3.1) is consistent with the nonlinear scheme (1.8).*

Proof. We have

$$V_h^{n+1} - V_h^n = \Delta t\, V_{ht}(\cdot, \theta^n),$$

with $t^n \le \theta^n \le t^{n+1}$. Thus

$$\|(V_h^{n+1} - V_h^n)^2\|_{L^2} = \|(\Delta t)^2(V_{ht}(\cdot, \theta_h^n))^2\|_{L^2} = (\Delta t)^2\|V_{ht}(\cdot, \theta_h^n)\|^2_{L^4}, \tag{3.28}$$

by (3.11). Again, by the fact that $H^1(\Omega)$ is continuously imbedded in $L^4(\Omega)$, [17,18], and the inverse estimate assumption (2.12), we have

$$\|(V_h^{n+1} - V_h^n)^2\|_{L^\infty(L^2)} \le C(\Delta t)^2h^{-2}\|V_{ht}\|^2_{L^\infty(L^2)}. \tag{3.29}$$

As before,

$$H''_\beta(v) = -\frac{k'_\beta(H_\beta(v))}{\left(k_\beta(H_\beta(v))\right)^3},$$

so that

$$|H''_\beta(v)| \le c\delta^{-3}, \tag{3.30}$$

by (2.5). Also

$$(f \circ H_\beta)''(v) = \frac{f''(H_\beta(v))k_\beta(H_\beta(v)) - f'(H_\beta(v))k'_\beta(H_\beta(v))}{\left(k_\beta(H_\beta(v))\right)^3}, \tag{3.31}$$

by (2.5). Hence

$$|(f \circ H_\beta)''(v)| \leq c\delta^{-3}. \tag{3.32}$$

Applying Lemma 3.3, and using (3.29), (3.30) and (3.32), we obtain

$$\left\| \frac{\left(V_h^{n+1} - V_h^n\right)^2}{2\Delta t} H''{}_\beta(\phi_h^n) \right\|_{L^\infty(L^2)} \leq c\,\Delta t\;\delta^{-9} h^{-6}, \tag{3.33}$$

and

$$\left\| \frac{\left(V_h^{n+1} - V_h^n\right)^2}{2} (f \circ H_\beta)''(\psi_h^n) \right\|_{L^\infty(L^2)} \leq c(\Delta t)^2\,\delta^{-9} h^{-6}. \tag{3.34}$$

This proves the theorem for fixed β and h. To get the uniform consistency in β and h, choose h and β in terms of Δt in such a way that the expression $\Delta t\;\delta^{-9} h^{-6}$ yields a positive fractional power of Δt. $\square$

Remark 3.5. As mentioned above, a balanced choice of β, h, and Δt is necessary to obtain an overall positive power of Δt in the proof above. It is known that $\delta \approx \beta^\mu$, with μ as in (1.3), for the regularized problem (2.7) (see [6]). For example, choose β and h such that $\beta \approx h^{\lambda_1}$ and $h \approx (\Delta t)^{\lambda_2}$ ($\lambda_1 > 0$ and $\lambda_2 > 0$), where by $u \approx v$, for $v \geq 0$ and $u \geq 0$, we mean $c_1 v \leq u \leq c_2 v$, for some positive constants c_1 and c_2. Then we need that

$$1 - 9\lambda_1\lambda_2\mu - 6\lambda_2 > 0.$$

For instance, if we want $(\Delta t)^{\frac{1}{2}}$, then we need

$$1 - 9\lambda_1\lambda_2\mu - 6\lambda_2 = \frac{1}{2}. \tag{3.35}$$

Solving for λ_2, we get

$$\lambda_2 = \frac{1}{6(3\lambda_1\mu + 2)}. \tag{3.36}$$

If we choose λ_1 as in (2.20), i.e.

$$\lambda_1 = \frac{4+\mu}{2 + 4\mu + \mu^2},$$

and substitute in (3.36), we obtain

$$\lambda_2 = \frac{2 + 4\mu + \mu^2}{6(2\mu^2 + 11\mu + 16)},$$

which is quite small but gives $\Delta t\;\delta^{-9} h^{-6} = O(\Delta t^{\frac{1}{2}})$.

3.2. A regularity result for the linearized scheme

We state and prove a regularity result which is a discrete analogue of Lemma 3.1 of [8], for Problem (3.1) and (3.2). It is also an analogue of Lemma 4.1 of [8], which was established for the nonlinear backward Euler scheme (1.8).

Lemma 3.6. *If $(U_h^n)_{n=0}^N \in M_h$ is the solution to the problem (3.1) and (3.2), then*

$$\sum_{0 \leq n \leq N-1} \Delta t \left\| \sqrt{H'_\beta(U_h^n)} \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|_{L^2}^2 + \eta \max_{0 \leq n \leq N-1} \left\| \nabla (U_h^{n+1}) \right\|_{L^2}^2 \leq C(\mathbf{u}) \tag{3.37}$$

for some $\eta > 0$, with $C(\mathbf{u}) = C(\mathbf{u}, U_h^0, \nabla U_h^0)$.

Proof. Set $\chi = U_h^{n+1} - U_h^n$ in (3.1) to obtain

$$\Delta t \left\| \sqrt{H'_\beta(U_h^n)} \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|_{L^2}^2 + \left\| \nabla U_h^{n+1} \right\|_{L^2}^2 - (\nabla U_h^{n+1}, \nabla U_h^n) = \Big(\big((f \circ H_\beta)(U_h^n) + \big(U_h^{n+1} - U_h^n\big) \times (f \circ H_\beta)'(U_h^n)\big)\,\mathbf{u}^{n+1}, \nabla\left(U_h^{n+1} - U_h^n\right)\Big). \tag{3.38}$$

Next using the fact that

$$-\left(\nabla U_h^{n+1}, \nabla U_h^n\right) \geq -\frac{1}{2}\left\|\nabla U_h^{n+1}\right\|_{L^2}^2 - \frac{1}{2}\left\|\nabla U_h^n\right\|_{L^2}^2,$$

(arithmetic–geometric inequality) we get

$$\Delta t \left\| \sqrt{H'_\beta(U_h^n)} \frac{U_h^{n+1} - U_h^n}{\Delta t} \right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla U_h^{n+1}\right\|_{L^2}^2 - \frac{1}{2}\left\|\nabla U_h^n\right\|_{L^2}^2 \leq (((f \circ H_\beta)(U_h^n) + (U_h^{n+1} - U_h^n)$$
$$\times (f \circ H_\beta)'(U_h^n))\mathbf{u}^{n+1}, \nabla(U_h^{n+1} - U_h^n)). \tag{3.39}$$

We treat each term on the righthand side of (3.39) separately. We have

$$\begin{aligned}\left((f \circ H_\beta)(U_h^n)\mathbf{u}^{n+1}, \nabla\left(U_h^{n+1} - U_h^n\right)\right) &= \left((f \circ H_\beta)\left(U_h^{n+1}\right)\mathbf{u}^{n+1}, \nabla U_h^{n+1}\right) - \left((f \circ H_\beta)\left(U_h^n\right)\mathbf{u}^n, \nabla U_h^n\right) \\ &\quad - \left(\left((f \circ H_\beta)\left(U_h^{n+1}\right) - (f \circ H_\beta)(U_h^n)\right)\mathbf{u}^{n+1}, \nabla\left(U_h^{n+1}\right)\right) \\ &\quad - \left((f \circ H_\beta)\left(U_h^n\right)\left(\mathbf{u}^{n+1} - \mathbf{u}^n\right), \nabla U_h^n\right).\end{aligned} \tag{3.40}$$

Now

$$\begin{aligned}\left|\left(f \circ H_\beta\right)'(U_h^n)\right| &= \left|f'\left(H_\beta\left(U_h^n\right)\right) H'_\beta(U_h^n)\right| \\ &= \left|f'\left(H_\beta\left(U_h^n\right)\right)\sqrt{H'_\beta(U_h^n)}\sqrt{H'_\beta(U_h^n)}\right| \\ &\leq \frac{\left|f'\left(H_\beta\left(U_h^n\right)\right)\right|}{\sqrt{k_\beta\left(H_\beta(U_h^n)\right)}}\sqrt{H'_\beta(U_h^n)} \leq C\sqrt{H'_\beta(U_h^n)},\end{aligned} \tag{3.41}$$

where we have used (1.7) and (2.15).

Using (3.41), we get

$$\begin{aligned}&\left|\left((U_h^{n+1} - U_h^n)(f \circ H_\beta)'(U_h^n)\right)\mathbf{u}^{n+1}, \nabla\left(U_h^{n+1} - U_h^n\right)\right| \\ &\quad\leq \frac{\sigma}{\Delta t}\left\|\sqrt{H'_\beta(U_h^n)}\left(U_h^{n+1} - U_h^n\right)\right\|_{L^2}^2 + c(\|\mathbf{u}\|_{L^\infty(L^\infty)})\frac{\Delta t}{\sigma}\left\|\nabla\left(U_h^{n+1} - U_h^n\right)\right\|_{L^2}^2 \\ &= \sigma\,\Delta t\left\|\sqrt{H'_\beta(U_h^n)}\left(\frac{U_h^{n+1} - U_h^n}{\Delta t}\right)\right\|_{L^2}^2 + c(\|\mathbf{u}\|_{L^\infty(L^\infty)})\frac{\Delta t}{\sigma}\left\|\nabla\left(U_h^{n+1} - U_h^n\right)\right\|_{L^2}^2\end{aligned} \tag{3.42}$$

where σ can be chosen as small as we wish (independently of β, h, and Δt) thanks to the arithmetic–geometric inequality. Since

$$\left\|\nabla\left(U_h^{n+1} - U_h^n\right)\right\|^2 \leq 2\left(\left\|\nabla U_h^{n+1}\right\|_{L^2}^2 + \left\|\nabla U_h^n\right\|_{L^2}^2\right), \tag{3.43}$$

estimate (3.40) and (3.42) in (3.39) yield

$$\begin{aligned}&\Delta t\left\|\sqrt{H'_\beta(U_h^n)}\frac{U_h^{n+1} - U_h^n}{\Delta t}\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla U_h^{n+1}\right\|_{L^2}^2 - \frac{1}{2}\left\|\nabla U_h^n\right\|_{L^2}^2 \\ &\quad\leq \left((f \circ H_\beta)\left(U_h^{n+1}\right)\mathbf{u}^{n+1}, \nabla U_h^{n+1}\right) - \left((f \circ H_\beta)\left(U_h^n\right)\mathbf{u}^n, \nabla U_h^n\right) \\ &\qquad - \left(\left((f \circ H_\beta)\left(U_h^{n+1}\right) - (f \circ H_\beta)(U_h^n)\right)\mathbf{u}^{n+1}, \nabla U_h^{n+1}\right) - \left((f \circ H_\beta)\left(U_h^n\right)\left(\mathbf{u}^{n+1} - \mathbf{u}^n\right), \nabla U_h^n\right) \\ &\qquad + \sigma\,\Delta t\left\|\sqrt{H'_\beta(U_h^n)}\left(\frac{U_h^{n+1} - U_h^n}{\Delta t}\right)\right\|_{L^2}^2 + c(\|\mathbf{u}\|_{L^\infty(L^\infty)})\frac{\Delta t}{\sigma}\left(\left\|\nabla U_h^{n+1}\right\|_{L^2}^2 + \left\|\nabla U_h^n\right\|_{L^2}^2\right).\end{aligned} \tag{3.44}$$

The third term on the right side of (3.44) can be bounded as follows.

$$\begin{aligned}&-\left(\left((f \circ H_\beta)\left(U_h^{n+1}\right) - (f \circ H_\beta)(U_h^n)\right)\mathbf{u}^{n+1}, \nabla U_h^{n+1}\right) \\ &\quad\leq \sigma_1\|\mathbf{u}\|_{L^\infty(L^\infty)}^2\Delta t\left\|(f \circ H_\beta)'(\theta_h^n)\left(\frac{U_h^{n+1} - U_h^n}{\Delta t}\right)\right\|_{L^2}^2 + \frac{\Delta t}{\sigma_1}\|\nabla U_h^{n+1}\|_{L^2}^2,\end{aligned} \tag{3.45}$$

where θ_h^n is between U_h^n and U_h^{n+1} i.e. $\theta_h^n = (1 - s_h)U_h^n + s_h U_h^{n+1}$ for some $0 \leq s_h \leq 1$.

We have

$$\begin{aligned}(f\circ H_\beta)'(\theta_h^n) &= f'(H_\beta(\theta_h^n))H'_\beta(\theta_h^n)\\ &= f'(H_\beta(\theta_h^n))\sqrt{H'(\theta_h^n)}\sqrt{H'(\theta_h^n)}\\ &= f'(H_\beta(\theta_h^n))\sqrt{H'(\theta_h^n)}\frac{\sqrt{H'(\theta_h^n)}}{\sqrt{H'(U_h^n)}}\sqrt{H'_\beta(U_h^n)}\\ &= \frac{f'(H_\beta(\theta_h^n))}{\sqrt{k_\beta(H_\beta(\theta_h^n))}}\frac{\sqrt{H'(\theta_h^n)}}{\sqrt{H'(U_h^n)}}\sqrt{H'_\beta(U_h^n)}. \end{aligned} \tag{3.46}$$

In view of (3.46), inequality (3.45) becomes

$$\begin{aligned}&\left|-\left(\left(\left(f\circ H_\beta\right)\left(U_h^{n+1}\right)-\left(f\circ H_\beta\right)\left(U_h^n\right)\right)\mathbf{u}^{n+1},\nabla U_h^{n+1}\right)\right|\\ &\quad\le c\sigma_2\sup_{0\le n\le N}\left|\frac{H'_\beta(\theta_h^n)}{H'_\beta(U_h^n)}\right|\Delta t\left\|\left(\sqrt{H'_\beta(U_h^n)}\frac{U_h^{n+1}-U_h^n}{\Delta t}\right)\right\|_{L^2}^2+\frac{\Delta t}{\sigma_2}\|\nabla U_h^{n+1}\|_{L^2}^2,\end{aligned} \tag{3.47}$$

where we have used (1.7) and (2.15). We note that, in the worse case we have:

$$\left|\frac{H'_\beta(\theta_h^n)}{H'_\beta(U_h^n)}\right|\le\frac{c}{\delta}, \tag{3.48}$$

with δ defined as in (2.3) and (2.4). Indeed, we have

$$\left|\frac{H'_\beta(\theta_h^n)}{H'_\beta(U_h^n)}\right|=\frac{k_\beta(H_\beta(U_h^n))}{k_\beta(H_\beta(\theta_h^n))}\le\frac{c}{\delta} \tag{3.49}$$

by (2.5) and (2.15). However we suspect this term to be bounded independently of β as would be the case for the nondegenerate problem. Indeed, if g is a regular function, the equality $g(y)-g(x)=g'(c)(y-x)$, for some c between x and y, and the fact $c\to x$ as $y\to x$, implies $g'(c)\to g'(x)$, if g is C^1. That is:

$$\frac{f'(c)}{f'(x)}\to 1$$

as $c\to x$. For this reason we will assume, for the present analysis, that

$$\left|\frac{H'_\beta(\theta_h^n)}{H'_\beta(U_h^n)}\right|\le C \tag{3.50}$$

independently of β, h, and Δt.

The fourth term on the right side of (3.44) can be bounded as follows.

$$\left|-\left(\left(f\circ H_\beta\right)\left(U_h^n\right)\left(\mathbf{u}^{n+1}-\mathbf{u}^n\right),\nabla U_h^n\right)\right|\le c\left\{\Delta t\left\|\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t}\right\|_{L^2}^2+\Delta t\|\nabla U_h^{n+1}\|_{L^2}^2\right\} \tag{3.51}$$

where we have used (2.14) and the fact that f is $C^1[0,1]$.

To finish the proof of the lemma, combine estimates (3.44) through (3.51), hide the appropriate terms in the left side (by choosing σ, σ_1, and σ_2 sufficiently small as allowed by the the arithmetic–geometric inequality), sum for $0\le n\le m-1$, for $1\le m\le N-1$, to get

$$\begin{aligned}&\sum_{0\le n\le m-1}\rho\Delta t\left\|\sqrt{H'_\beta(U_h^n)}\frac{U_h^{n+1}-U_h^n}{\Delta t}\right\|_{L^2}^2+\frac{1}{2}\|\nabla U_h^m\|_{L^2}^2-\frac{1}{2}\|\nabla U_h^0\|_{L^2}^2\\ &\quad\le((f\circ H_\beta)(U_h^m)\mathbf{u}^m,\nabla U_h^m)-((f\circ H_\beta)(U_h^0)\mathbf{u}^0,\nabla U_h^0)+C(\mathbf{u})\Delta t\sum_{0\le n\le m-1}\|\nabla U_h^n\|_{L^2}^2+c\Delta t\|\nabla U_h^m\|_{L^2}^2\end{aligned} \tag{3.52}$$

for some $\rho>0$. The first term on the right side of (3.52) is bounded as follows.

$$\begin{aligned}|((f(H_\beta(U_h^m))\mathbf{u}^m,\nabla U_h^m))| &\le c\|f(H_\beta(U_h^m))\mathbf{u}^m\|_{L^2}^2+\frac{1}{4}\|\nabla U_h^m\|_{L^2}^2\\ &\le C(\mathbf{u})+\frac{1}{4}\|\nabla U_h^m\|_{L^2}^2,\end{aligned} \tag{3.53}$$

by (2.14) and the fact that f is bounded on $[1, 0]$. We can hide the respective last terms on the right hand sides of (3.52) and (3.53) in the left side of (3.52), and obtain

$$\sum_{0\le n\le m-1} \rho\Delta t \left\| \sqrt{H'_\beta(U_h^n)}\frac{U_h^{n+1}-U_h^n}{\Delta t}\right\|_{L^2}^2 + \left(\frac{1}{4} - c\Delta t\right)\|\nabla U_h^m\|_{L^2}^2$$
$$\le \tilde{C}(\mathbf{u})\Delta t \sum_{0\le n\le m-1} \|\nabla U_h^n\|_{L^2}^2 + C(\mathbf{u}) + \frac{1}{2}\|\nabla U_h^0\|_{L^2}^2. \tag{3.54}$$

Let Δt be so small that

$$\frac{1}{4} - c\Delta t > c_0 > 0.$$

Finally, use the Discrete Gronwall Inequality (see, for instance, Lemma 4.3 of [19]) and take the sup for $1 \le m \le N$ to get the Lemma. $\square$

Remark 3.7. This work has focused on the degenerate case i.e. on the case $k(0) = k(1) = 0$. Nevertheless scheme (3.1) and (3.2) can also be applied to the nondegenerate case $k(s) > k_0 > 0$. In this case, (3.37) is valid without the strong condition (3.49), since then $\delta \ge c_0 > 0$.

4. Conclusion

We have established, in this paper, a regularity result which has helped to show that the linearized scheme (3.1) proposed in [7] is consistent with the nonlinear scheme (1.8) analyzed in [8]. This is done without pretending to any optimal estimates. In the sequel of the work, we intend to establish effective error estimates for the linearized scheme and an effective numerical computation of the solution of the scheme with a numerical visualization of the solution for problem (1.1). We believe that there is room for improvement for the estimates established here.

The present analysis relies on condition (1.6) which implies a diffusion-dominant flow. For the general case, we believe an investigation should combine localized methods, like ELLAM (see for example [20–22]) and linearized methods (where the nonlinear data are linearized in some way). We believe the ideas in the present work could be useful.

As mentioned above, we notice that scheme (3.1) and (3.2) can also be applied to a nondegenerate problem, in which case the competition in (3.33) and (3.34) is only between the spatial discretization parameter h and the time stepping parameter Δt.

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